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Spin 1 particle on 4-dimensional sphere: extended helicity operator, separation of the variables, and exact solutions

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Abstract

Spin 1 particle is investigated in 3-dimensional curved space of constant positive curvature. An extended helicity operator is defined and the variables are separated in a tetrad-based 10-dimensional Duffin-Kemmer equation in quasi cylindrical coordinates. The problem is solved exactly in hypergeometric functions, the energy spectrum determined by three discrete quantum numbers is obtained. Transition to a massless case of electromagnetic field is performed.

In 3-dimensional spherical Riemann space S_3 will use the following system of quasi-cylindric coordinates (see [1]; the same coordinate system was used when treating Landau problem in 3-dimensional spaces of constant curvature in [4, 5, 6])

$$\begin{aligned} dS^2 &= c^2 dt^2 - \rho^2 [\cos^2 z (dr^2 + \sin^2 r d\phi^2) + dz^2] , \\ z &\in [-\pi/2, +\pi/2] , \quad r \in [0, +\pi] , \quad \phi \in [0, 2\pi] ; \end{aligned} \quad (1)$$

a diagonal tetrad (let $x^\alpha = (t, r, \phi, z)$)

$$e_{(a)}^\beta(x) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos^{-1} z & 0 & 0 \\ 0 & 0 & \cos^{-1} z \sin^{-1} r & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} ; \quad (2)$$

corresponding Christofel ant Ricci coefficients are

$$\begin{aligned}
\Gamma^r_{jk} &= \begin{vmatrix} 0 & 0 & -\tan z \\ 0 & -\sin r \cos r & 0 \\ -\tan z & 0 & 0 \end{vmatrix}, \\
\Gamma^\phi_{jk} &= \begin{vmatrix} 0 & \cot r & 0 \\ \cot r & 0 & -\tan z \\ 0 & -\tan z & 0 \end{vmatrix}, \\
\Gamma^z_{jk} &= \begin{vmatrix} \sin z \cos z & 0 & 0 \\ 0 & \sin z \cos z \sin^2 r & 0 \\ 0 & 0 & 0 \end{vmatrix}, \\
\gamma_{122} &= \frac{1}{\cos z \tan r}, \quad \gamma_{311} = -\tan z, \quad \gamma_{322} = -\tan z. \quad (3)
\end{aligned}$$

Tetrad-based Duffin-Kemmer equation (the notation from [2] is used) takes the form

$$\begin{aligned}
&\left\{ i\beta^0 \frac{\partial}{\partial t} + \frac{1}{\cos z} \left(i\beta^1 \frac{\partial}{\partial r} + \beta^2 \frac{i\partial_\phi + iJ^{12} \cos r}{\sin r} \right) \right. \\
&\left. + i\beta^3 \frac{\partial}{\partial z} + i \frac{\sin z}{\cos z} (\beta^1 J^{13} + \beta^2 J^{23}) - M \right\} \Psi = 0. \quad (4)
\end{aligned}$$

To separate the variables, we take the substitution

$$\Psi = e^{-i\epsilon t} e^{im\phi} \begin{vmatrix} \Phi_0(r, z) \\ \vec{\Phi}(r, z) \\ \vec{E}(r, z) \\ \vec{H}(r, z) \end{vmatrix} \quad (5)$$

and use a block-representation (we use so-called cyclic basis for 10×10

Duffin-Kemmer matrices – see in [3])

$$\begin{aligned}
& \left[\epsilon \cos z \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} + i \begin{vmatrix} 0 & 0 & e_1 & 0 \\ 0 & 0 & 0 & \tau_1 \\ -e_1^+ & 0 & 0 & 0 \\ 0 & -\tau_1 & 0 & 0 \end{vmatrix} \frac{\partial}{\partial r} \right. \\
& \quad - \frac{1}{\sin r} \begin{vmatrix} 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & \tau_2 \\ -e_2^+ & 0 & 0 & 0 \\ 0 & -\tau_2 & 0 & 0 \end{vmatrix} (\nu - \cos r S_3) \\
& \quad \left. + i \cos z \begin{vmatrix} 0 & 0 & e_3 & 0 \\ 0 & 0 & 0 & \tau_3 \\ -e_3^+ & 0 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 \end{vmatrix} \frac{\partial}{\partial z} \right. \\
& \quad \left. + i \sin z \begin{vmatrix} 0 & 0 & -2e_3 & 0 \\ 0 & 0 & 0 & -\tau_3 \\ 0 & 0 & 0 & 0 \\ 0 & +\tau_3 & 0 & 0 \end{vmatrix} - M \cos z \right] \begin{vmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{vmatrix} = 0, \quad (6)
\end{aligned}$$

or

$$\begin{aligned}
& ie_1 \partial_r \vec{E} - \frac{1}{\sin r} e_2 (m - \cos r s_3) \vec{E} + \\
& + i(\cos z \partial_z - 2 \sin z) e_3 \vec{E} = M \cos z \Phi_0, \\
& i \epsilon \cos z \vec{E} + i \tau_1 \partial_r \vec{H} - \frac{\tau_2}{\sin r} (m - \cos r s_3) \vec{H} + \\
& + i(\cos z \partial_z - \sin z) \tau_3 \vec{H} = M \cos z \vec{\Phi}, \\
& -i \epsilon \cos z \vec{\Phi} - i e_1^+ \partial_r \Phi_0 + \frac{m}{\sin r} e_2^+ \Phi_0 - i \cos z e_3^+ \partial_z \Phi_0 = M \cos z \vec{E}, \\
& -i \tau_1 \partial_r \vec{\Phi} + \frac{(m - \cos r s_3)}{\sin r} \tau_2 \vec{\Phi} - i(\cos z \partial_z - \\
& - i \sin z) \tau_3 \vec{\Phi} = M \cos z \vec{H}. \quad (7)
\end{aligned}$$

After calculations needed we arrive at the system

$$\begin{aligned}
& \gamma \left(\frac{\partial E_1}{\partial r} - \frac{\partial E_3}{\partial r} \right) - \frac{\gamma}{\sin r} [(m - \cos r) E_1 + (m + \cos r) E_3] - \\
& - (\cos z \frac{\partial}{\partial z} - 2 \sin z) E_2 = M \cos z \Phi_0, \quad (8)
\end{aligned}$$

$$\begin{aligned}
& + i\epsilon \cos z E_1 + i\gamma \frac{\partial H_2}{\partial r} + i\gamma \frac{m}{\sin r} H_2 + \\
& + i(\cos z \frac{\partial}{\partial z} - \sin z) H_1 = M \cos z \Phi_1 , \\
& + i\epsilon \cos z E_2 + i\gamma (\frac{\partial H_1}{\partial r} + \frac{\partial H_3}{\partial r}) - \\
& - \frac{i\gamma}{\sin r} [(m - \cos r) H_1 - (m + \cos r) H_3] = M \cos z \Phi_2 , \\
& + i\epsilon \cos z E_3 + i\gamma \frac{\partial H_2}{\partial r} - i\gamma \frac{m}{\sin r} H_2 - \\
& - i(\cos z \frac{\partial}{\partial z} - \sin z) H_3 = M \cos z \Phi_3 , \tag{9}
\end{aligned}$$

$$\begin{aligned}
& - i\epsilon \cos z \Phi_1 + \gamma \frac{\partial \Phi_0}{\partial r} + \gamma \frac{m}{\sin r} \Phi_0 = M \cos z E_1 , \\
& - i\epsilon \Phi_2 - \frac{\partial \Phi_0}{\partial z} = M E_2 , \\
& - i\epsilon \cos z \Phi_3 - \gamma \frac{\partial \Phi_0}{\partial r} + \gamma \frac{m}{\sin r} \Phi_0 = M \cos z E_3 , \tag{10}
\end{aligned}$$

$$\begin{aligned}
& - i\gamma \frac{\partial \Phi_2}{\partial r} - i\gamma \frac{m}{\sin r} \Phi_2 - i(\cos z \frac{\partial}{\partial z} - \sin z) \Phi_1 = M \cos z H_1 , \\
& - i\gamma (\frac{\partial \Phi_1}{\partial r} + \frac{\partial \Phi_3}{\partial r}) + \frac{i\gamma}{\sin r} [(m - \cos r) \Phi_1 - (m + \cos r) \Phi_3] = M \cos z H_2 , \\
& - i\gamma \frac{\partial \Phi_2}{\partial r} + i\gamma \frac{m}{\sin r} \Phi_2 + i(\cos z \frac{\partial}{\partial z} - \sin z) \Phi_3 = M \cos z H_3 . \tag{11}
\end{aligned}$$

With the use of the notation

$$\begin{aligned}
& \gamma (\frac{\partial}{\partial r} + \frac{m - \cos r}{\sin r}) = a_- , \\
& \gamma (\frac{\partial}{\partial r} + \frac{m + \cos r}{\sin r}) = a_+ , \\
& \gamma (\frac{\partial}{\partial r} + \frac{m}{\sin r}) = a , \\
& \gamma (-\frac{\partial}{\partial r} + \frac{m - \cos r}{\sin r}) = b_- , \\
& \gamma (-\frac{\partial}{\partial r} + \frac{m + \cos r}{\sin r}) = b_+ , \\
& \gamma (-\frac{\partial}{\partial r} + \frac{m}{\sin r}) = b , \tag{12}
\end{aligned}$$

it reads simpler

$$-b_- E_1 - a_+ E_3 - \cos z \left(\frac{\partial}{\partial z} - 2 \tan z \right) E_2 = M \cos z \Phi_0, \quad (13)$$

$$\begin{aligned} ia H_2 + i\epsilon \cos z E_1 + i \cos z \left(\frac{\partial}{\partial z} - \tan z \right) H_1 &= M \cos z \Phi_1, \\ -ib_- H_1 + ia_+ H_3 + i\epsilon \cos z E_2 &= M \cos z \Phi_2, \\ -ib H_2 + i\epsilon \cos z E_3 - i \left(\frac{\partial}{\partial z} - \tan z \right) H_3 &= M \cos z \Phi_3, \end{aligned} \quad (14)$$

$$\begin{aligned} a \Phi_0 - i\epsilon \cos z \Phi_1 &= M \cos z E_1, \\ -i\epsilon \Phi_2 - \frac{\partial}{\partial z} \Phi_0 &= M E_2, \\ b \Phi_0 - i\epsilon \cos z \Phi_3 &= M \cos z E_3, \end{aligned} \quad (15)$$

$$\begin{aligned} -ia \Phi_2 - i \cos z \left(\frac{\partial}{\partial z} - \tan z \right) \Phi_1 &= M \cos z H_1, \\ i b_- \Phi_1 - ia_+ \Phi_3 &= M \cos z H_2, \\ i b \Phi_2 + i \cos \left(\frac{\partial}{\partial z} - \tan z \right) \Phi_3 &= M \cos H_3. \end{aligned} \quad (16)$$

Let us employ additional operator, a generalized helicity operator – such that

$$\Sigma \Psi = \sigma \Psi, \quad \Psi = e^{-i\epsilon t} e^{im\phi} \begin{pmatrix} \Phi_0(r, z) \\ \vec{\Phi}(r, z) \\ \vec{E}(r, z) \\ \vec{H}(r, z) \end{pmatrix},$$

$$\left[\frac{1}{\cos z} \left(S_1 \frac{\partial}{\partial r} + i S_2 \frac{m - S_3 \cos r}{\sin r} \right) + \left(\frac{\partial}{\partial z} - \tan z \right) S_3 \right] \begin{pmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{pmatrix} = \sigma \begin{pmatrix} \Phi_0 \\ \vec{\Phi} \\ \vec{E} \\ \vec{H} \end{pmatrix}. \quad (17)$$

From (17) it follows the system of 10 equations (let $\gamma = 1/\sqrt{2}$):

$$0 = \sigma \Phi_0, \quad (18)$$

$$\begin{aligned}
\gamma \frac{\partial}{\partial r} \Phi_2 + \gamma \frac{m}{\sin r} \Phi_2 + \cos z \left(\frac{\partial}{\partial z} - \tan z \right) \Phi_1 &= \sigma \cos z \Phi_1 , \\
\gamma \left(\frac{\partial}{\partial r} \Phi_1 + \frac{\partial}{\partial r} \Phi_3 \right) - \\
-\frac{\gamma}{\sin r} [(m - \cos r) \Phi_1 - (m + \cos r) \Phi_3] &= \sigma \cos z \Phi_2 , \\
\gamma \frac{\partial}{\partial r} \Phi_2 - \gamma \frac{m}{\sin r} \Phi_2 - \cos z \left(\frac{\partial}{\partial z} - \tan z \right) \Phi_3 &= \sigma \cos z \Phi_3 , \tag{19}
\end{aligned}$$

$$\begin{aligned}
\gamma \frac{\partial}{\partial r} E_2 + \gamma \frac{m}{\sin r} E_2 + \cos z \left(\frac{\partial}{\partial z} - \tan z \right) E_1 &= \sigma \cos z E_1 , \\
\gamma \left(\frac{\partial}{\partial r} E_1 + \frac{\partial}{\partial r} E_3 \right) - \\
-\frac{\gamma}{\sin r} [(m - \cos r) E_1 - (m + \cos r) E_3] &= \sigma \cos z E_2 , \\
\gamma \frac{\partial}{\partial r} E_2 - \gamma \frac{m}{\sin r} E_2 - \cos z \left(\frac{\partial}{\partial z} - \tan z \right) E_3 &= \sigma \cos z E_3 , \tag{20}
\end{aligned}$$

$$\begin{aligned}
\gamma \frac{\partial}{\partial r} H_2 + \gamma \frac{m}{\sin r} H_2 + \cos z \left(\frac{\partial}{\partial z} - \tan z \right) H_1 &= \sigma \cos z H_1 , \\
\gamma \left(\frac{\partial}{\partial r} H_1 + \frac{\partial}{\partial \sin r} H_3 \right) - \\
-\frac{\gamma}{\sin r} [(m - \cos r) H_1 - (m + \cos r) H_3] &= \sigma \cos z H_2 , \\
\gamma \frac{\partial}{\partial r} H_2 - \gamma \frac{m}{\sin r} H_2 - \cos z \left(\frac{\partial}{\partial z} - \tan z \right) H_3 &= \sigma \cos z H_3 . \tag{21}
\end{aligned}$$

With notation (12) it reads simpler

$$0 = \sigma \Phi_0 , \tag{22}$$

$$\begin{aligned}
+ \cos z \left(\frac{\partial}{\partial z} - \tan z \right) \Phi_1 &= \sigma \cos z \Phi_1 - a \Phi_2 , \\
-b_- \Phi_1 + a_+ \Phi_3 &= \sigma \cos z \Phi_2 , \\
- \cos z \left(\frac{\partial}{\partial z} - \tan z \right) \Phi_3 &= \sigma \cos z \Phi_3 + b \Phi_2 , \tag{23}
\end{aligned}$$

$$\begin{aligned}
+ \cos z \left(\frac{\partial}{\partial z} - \tan z \right) E_1 &= \sigma \cos z E_1 - a E_2 , \\
-b_- E_1 + a_+ E_3 &= \sigma \cos z E_2 , \\
- \cos z \left(\frac{\partial}{\partial z} - \tan z \right) E_3 &= \sigma \cos z E_3 + b E_2 , \tag{24}
\end{aligned}$$

$$\begin{aligned}
& + \cos z \left(\frac{\partial}{\partial z} - \tan z \right) H_1 = \sigma \cos z H_1 - a H_2 , \\
& \quad -b_- H_1 + a_+ H_3 = \sigma \cos z H_2 , \\
& - \cos z \left(\frac{\partial}{\partial z} - \tan z \right) H_3 = \sigma \cos z H_3 + b H_2 .
\end{aligned} \tag{25}$$

Taking into account eqs. (22) – (25), from (13) – (16) we get

$$-b_- E_1 - a_+ E_3 - \cos z \left(\frac{\partial}{\partial z} - 2 \tan z \right) E_2 = M \cos z \Phi_0 , \tag{26}$$

$$\begin{aligned}
i\epsilon E_1 + i\sigma H_1 &= M \Phi_1 , \\
i\sigma H_2 + i\epsilon E_2 &= M \Phi_2 , \\
i\epsilon E_3 + i\sigma H_3 &= M \Phi_3 ,
\end{aligned} \tag{27}$$

$$\begin{aligned}
a \Phi_0 - i\epsilon \cos z \Phi_1 &= M \cos z E_1 , \\
-i\epsilon \Phi_2 - \frac{\partial}{\partial z} \Phi_0 &= M E_2 , \\
b \Phi_0 - i\epsilon \cos z \Phi_3 &= M \cos z E_3 ,
\end{aligned} \tag{28}$$

$$\begin{aligned}
-\sigma \Phi_1 &= M H_1 , \\
-i\sigma \Phi_2 &= M H_2 , \\
-i\sigma \Phi_3 &= M H_3 .
\end{aligned} \tag{29}$$

Below we will need an explicit form of the Lorentz condition. Starting from its tensor form

$$\begin{aligned}
& \nabla_\beta (e^{(b)\beta} \Phi_{(b)}^{cart}) = 0 \quad \implies \\
& \frac{\partial \Phi_{(b)}^{cart}}{\partial x^\beta} e^{(b)\beta} + \Phi_{(b)}^{cart} \nabla_\beta e^{(b)\beta} = 0 ,
\end{aligned} \tag{30}$$

or

$$\frac{\partial \Phi_{(b)}^{cart}}{\partial x^\beta} e^{(b)\beta} + \Phi_{(b)}^{cart} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\beta} \sqrt{-g} e^{(b)\beta} = 0 , \tag{31}$$

and taking into consideration (1)–(2), we transform eq. (31) to the form

$$\begin{aligned}
& \frac{\partial}{\partial t} \Phi_0^{cart} - \frac{1}{\cos z} \frac{\partial}{\partial r} \Phi_1^{cart} - \frac{1}{\cos z \sin r} \frac{\partial}{\partial \phi} \Phi_2^{cart} - \frac{\partial}{\partial z} \Phi_3^{cart} - \\
& - \Phi_1^{cart} \frac{1}{\cos^2 z \sin r} \frac{\partial}{\partial r} \cos^2 z \sin r \frac{1}{\cos z} - \Phi_3^{cart} \frac{1}{\cos^2 z \sin r} \frac{\partial}{\partial z} \cos^2 z \sin r = 0 ,
\end{aligned}$$

that is

$$\begin{aligned} & \frac{\partial}{\partial t} \Phi_0^{cart} - \frac{1}{\cos z} \left(\frac{\partial}{\partial r} + \frac{\cos r}{\sin r} \right) \Phi_1^{cart} - \\ & - \frac{1}{\cos z \sin r} \frac{\partial}{\partial \phi} \Phi_2^{cart} - \left(\frac{\partial}{\partial z} - 2 \tan z \right) \Phi_3^{cart} = 0 . \end{aligned}$$

From whence with the substitution (5) we obtain

$$\begin{aligned} & -i\epsilon \Phi_0^{cart} - \frac{1}{\cos z} \left(\frac{\partial}{\partial r} + \frac{\cos r}{\sin r} \right) \Phi_1^{cart} - \\ & - \frac{im}{\cos z \sin r} \Phi_2^{cart} - \left(\frac{\partial}{\partial z} - 2 \tan z \right) \Phi_3^{cart} = 0 . \end{aligned} \quad (32)$$

To use this relation in the above equations, we should transform (32) to cyclic basis:

$$\begin{aligned} \Phi_0 &= \Phi_0^{cart} , & \Phi_2 &= \Phi_3^{cart} , \\ \Phi_3 - \Phi_1 &= \sqrt{2} \Phi_1^{cart} , & \Phi_3 + \Phi_1 &= \sqrt{2}i \Phi_2^{cart} ; \end{aligned}$$

thus we have

$$\begin{aligned} & -i\epsilon \Phi_0 - \frac{1}{\cos z} \left(\frac{\partial}{\partial r} + \frac{\cos r}{\sin r} \right) \frac{\Phi_3 - \Phi_1}{\sqrt{2}} - \\ & - \frac{im}{\cos z \sin r} \frac{\Phi_3 + \Phi_1}{\sqrt{2}i} - \left(\frac{\partial}{\partial z} - 2 \tan z \right) \Phi_2 = 0 , \end{aligned} \quad (33)$$

that is

$$-i\epsilon \Phi_0 - \frac{1}{\cos z} b_- \Phi_1 - \frac{1}{\cos z} a_+ \Phi_3 - \left(\frac{\partial}{\partial z} - 2 \tan z \right) \Phi_2 = 0 . \quad (34)$$

Now, let us turn to eqs. (26) – (29). First, let us consider the case $\sigma \neq 0$, when one must accept from the very beginning such a restriction $\Phi_0 = 0$; correspondingly, equation become more simple

$$\begin{aligned} & \sigma \neq 0 , & \Phi_0 &= 0 , \\ & -b_- E_1 - a_+ E_3 - \cos z \left(\frac{\partial}{\partial z} - 2 \tan z \right) E_2 = 0 , \end{aligned} \quad (35)$$

$$\begin{aligned} i\epsilon E_1 + i\sigma H_1 &= M \Phi_1 , \\ i\sigma H_2 + i\epsilon E_2 &= M \Phi_2 , \\ i\epsilon E_3 + i\sigma H_3 &= M \Phi_3 , \end{aligned} \quad (36)$$

$$\begin{aligned}
-i\epsilon \Phi_1 &= M E_1 , \\
-i\epsilon \Phi_2 &= M E_2 , \\
-i\epsilon \Phi_3 &= M E_3 ,
\end{aligned} \tag{37}$$

$$\begin{aligned}
-\sigma \Phi_1 &= M H_1 , \\
-i\sigma \Phi_2 &= M H_2 , \\
-i\sigma \Phi_3 &= M H_3 .
\end{aligned} \tag{38}$$

Note that substituting (33) into (30), one gets

$$-b_- \Phi_1 - a_+ \Phi_3 - \cos z \left(\frac{\partial}{\partial z} - 2 \tan z \right) \Phi_2 = 0 , \tag{39}$$

which coincides with the Lorentz condition (34) when $\Phi_0 = 0$. Condition (39) can be simplified by the following substitutions

$$\Phi_1 = \frac{\varphi_1}{\cos z} , \quad \Phi_3 = \frac{\varphi_3}{\cos z} , \quad \Phi_2 = \frac{1}{\cos^2 z} \varphi_2 ,$$

which results in

$$-b_- \varphi_1 - a_+ \varphi_3 - \frac{\partial}{\partial z} \varphi_2 = 0 ; \tag{40}$$

in new variables $b_- \varphi_1 = \bar{\varphi}_1$, $a_+ \varphi_3 = \bar{\varphi}_3$ it becomes yet simpler

$$\bar{\varphi}_1 + \bar{\varphi}_3 + \frac{\partial}{\partial z} \varphi_2 = 0 . \tag{41}$$

Remaining algebraic relations will fixe values of σ and relative coefficients of various components

$$\begin{aligned}
\sigma &= \pm i \sqrt{\epsilon^2 - M^2} , \quad \Phi_0 = 0 , \\
H_j &= -i \frac{\sigma}{M} \Phi_j , \quad E_j = \frac{i\epsilon}{M} \Phi_j .
\end{aligned} \tag{42}$$

Explicit form of the main functions Φ_j will be found below when exploring helicity operator equations.

In masless case instead of (42) we have

$$\begin{aligned}
\sigma &= \pm i\epsilon , \quad \Phi_0 = 0 , \\
H_j &= -i\sigma \Phi_j , \quad E_j = i\epsilon \Phi_j .
\end{aligned} \tag{43}$$

Now, let us consider the case $\sigma = 0$, when the system (26) – (29) is

$$-b_- E_1 - a_+ E_3 - \cos z \left(\frac{\partial}{\partial z} - 2 \tan z \right) E_2 = M \cos z \Phi_0 , \quad (44)$$

$$i\epsilon E_1 = M \Phi_1 , \quad i\epsilon E_2 = M \Phi_2 , \quad i\epsilon E_3 = M \Phi_3 , \quad (45)$$

$$\begin{aligned} a \Phi_0 - i\epsilon \cos z \Phi_1 &= M \cos z E_1 , \\ -i\epsilon \Phi_2 - \frac{\partial}{\partial z} \Phi_0 &= M E_2 , \\ b \Phi_0 - i\epsilon \cos z \Phi_3 &= M \cos z E_3 , \end{aligned} \quad (46)$$

$$0 = M H_1 , \quad 0 = M H_2 , \quad 0 = M H_3 . \quad (47)$$

Note that allowing for (45), from (44) it follows

$$-b_- \Phi_1 - a_+ \Phi_3 - \cos z \left(\frac{\partial}{\partial z} - 2 \tan z \right) \Phi_2 = i\epsilon \cos z \Phi_0 , \quad (48)$$

which coincides with the Lorentz condition.

Let us introduce substitutions

$$\begin{aligned} \Phi_1 &= \frac{\varphi_1}{\cos z} , & \Phi_3 &= \frac{\varphi_3}{\cos z} , & \Phi_2 &= \frac{1}{\cos^2 z} \varphi_2 , \\ E_1 &= \frac{e_1}{\cos z} , & E_3 &= \frac{e_3}{\cos z} , & E_2 &= \frac{1}{\cos^2 z} e_2 , \\ b_- \varphi_1 &= \bar{\varphi}_1 , & a_+ \varphi_3 &= \bar{\varphi}_3 , & b_- e_1 &= \bar{e}_1 , & a_+ e_3 &= \bar{e}_3 , \end{aligned}$$

then eqs. (44) – (47) read

$$\bar{\varphi}_1 + \bar{\varphi}_3 + \frac{\partial}{\partial z} \varphi_2 = -i\epsilon \cos^2 z \Phi_0 , \quad (49)$$

$$i\epsilon e_1 = M \varphi_1 , \quad i\epsilon e_2 = M \varphi_2 , \quad i\epsilon e_3 = M \varphi_3 , \quad (50)$$

$$\begin{aligned} \Delta \Phi_0 - i\epsilon \bar{\varphi}_1 &= M \bar{e}_1 , \\ -i\epsilon \varphi_2 - \cos^2 z \frac{\partial}{\partial z} \Phi_0 &= M e_2 , \\ \Delta \Phi_0 - i\epsilon \bar{\varphi}_3 &= M \bar{e}_3 , \end{aligned} \quad (51)$$

$$0 = H_1 , \quad 0 = H_2 , \quad 0 = H_3 , \quad (52)$$

one should take into consideration identity $\Delta = b_- a = a_+ b$.

Below we will show from helicity operator eigenvalue equation that when $\sigma = 0$ there must hold the following relationships

$$\begin{aligned} \bar{\varphi}_1 = \bar{\varphi}_3 = \bar{\varphi}, \quad \bar{e}_1 = \bar{e}_3 = \bar{e}, \quad \bar{h}_1 = \bar{h}_3 = \bar{h}, \\ \Delta\varphi_2 = -\cos^2 z \frac{\partial}{\partial z} \bar{\varphi}, \quad \Delta e_2 = -\cos^2 z \frac{\partial}{\partial z} \bar{e}, \quad \Delta h_2 = -\cos^2 z \frac{\partial}{\partial z} \bar{h}; \end{aligned} \quad (53)$$

so that from (49) – (52) we get

$$-2\bar{\varphi} - \frac{\partial}{\partial z} \varphi_2 = i\epsilon \cos^2 z \Phi_0, \quad (54)$$

$$\bar{e} = \frac{M}{i\epsilon} \bar{\varphi}, \quad e_2 = \frac{M}{i\epsilon} \varphi_2, \quad H_j = 0, \quad (55)$$

$$\begin{aligned} (\epsilon^2 - M^2)\varphi_2 - i\epsilon \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0, \\ i\epsilon \Delta \Phi_0 + (\epsilon^2 - M^2)\bar{\varphi} = 0. \end{aligned} \quad (56)$$

Acting on the first equation in (56) by the operator ∂_z , and excluding in second equation in (56) the variable $\bar{\varphi}$ with the help of (54) – thus we get

$$\begin{aligned} (\epsilon^2 - M^2) \frac{\partial}{\partial z} \varphi_2 - i\epsilon \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0, \\ 2i\epsilon \Delta \Phi_0 - (\epsilon^2 - M^2) \left(\frac{\partial}{\partial z} \varphi_2 + i\epsilon \cos^2 z \Phi_0 \right) = 0. \end{aligned}$$

Summing these two equations, we arrive at a second order equation for Φ_0

$$-i\epsilon \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} \Phi_0 + 2i\epsilon \Delta \Phi_0 - (\epsilon^2 - M^2) i\epsilon \cos^2 z \Phi_0 = 0,$$

that is

$$\left(-2\Delta + \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} + (\epsilon^2 - M^2) \cos^2 z \right) \Phi_0 = 0. \quad (57)$$

In eq. (57), the variables are separated straightforwardly

$$\begin{aligned} \Phi_0(r, z) = \Phi_0(r) \Phi_0(z), \quad \frac{1}{\Phi_0(r)} (2\Delta) \Phi_0(r) = \Lambda, \\ \frac{1}{\Phi_0(z)} \left(\frac{d}{dz} \cos^2 z \frac{d}{dz} + (\epsilon^2 - M^2) \cos^2 z \right) \Phi_0(z) = \Lambda. \end{aligned} \quad (58)$$

In the same manner, with the help of (54) one can exclude the function Φ_0 from second equation in (56)

$$\Delta(-2\bar{\varphi} - \frac{\partial}{\partial z}\varphi_2) + (\epsilon^2 - M^2) \cos^2 z \bar{\varphi} = 0 ,$$

and further excluding the variable $\Delta\varphi_2$ with the help of (53) we arrive at a second order equation for $\bar{\varphi}$

$$\left(-2\Delta + \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} + (\epsilon^2 - M^2) \cos^2 z \right) \bar{\varphi} = 0 . \quad (59)$$

In this equation, the variables are separated as well

$$\begin{aligned} \bar{\varphi}(r, z) &= \bar{\varphi}(r) \bar{\varphi}(z) , \\ \frac{1}{\bar{\varphi}(r)} (2\Delta) \bar{\varphi}(r) &= \Lambda , \\ \frac{1}{\bar{\varphi}(z)} \left(\frac{d}{dz} \cos^2 z \frac{d}{dz} + (\epsilon^2 - M^2) \cos^2 z \right) \bar{\varphi}(z) &= \Lambda . \end{aligned} \quad (60)$$

Note, that from the first equation in (56) it follows an expression for φ_2

$$\varphi_2 = \frac{i\epsilon \cos^2 z}{(\epsilon^2 - M^2)} \frac{\partial}{\partial z} \Phi_0 . \quad (61)$$

One can easily verify consistency of the relations obtained. Indeed, let us act on eq. (61) by the operator Δ

$$\Delta\varphi_2 = \frac{i\epsilon}{(\epsilon^2 - M^2)} \Delta \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0 .$$

Further, allowing for (53) we get

$$-\cos^2 z \frac{\partial}{\partial z} \bar{\varphi} = \frac{i\epsilon}{(\epsilon^2 - M^2)} \Delta \cos^2 z \frac{\partial}{\partial z} \Phi_0 = 0 ,$$

from whence it follows

$$i\epsilon \Delta \Phi_0 + (\epsilon^2 - M^2) \bar{\varphi} = 0 ,$$

which is an identity

$$-\frac{\partial}{\partial z} \bar{\varphi} \equiv -\frac{1}{(\epsilon^2 - M^2)} (\epsilon^2 - M^2) \frac{\partial}{\partial z} \bar{\varphi} .$$

Now, let us turn to equations steaming from diagonalization of helicity operator. In (22) – (25) owe can notice three similar groups of equations. For instance, equations for H_i are

$$\begin{aligned} a H_2 + \cos z \left(\frac{\partial}{\partial z} - \tan z \right) H_1 &= \sigma \cos z H_1 , \\ -b_- H_1 + a_+ H_3 &= \sigma \cos z H_2 , \\ -b H_2 - \cos z \left(\frac{\partial}{\partial z} - \tan z \right) H_3 &= \sigma \cos z H_3 . \end{aligned} \quad (62)$$

With the help of substitutions

$$H_1 = \frac{1}{\cos z} h_1(r, z) , \quad H_2 = \frac{1}{\cos^2 z} h_2(r, z) , \quad H_3 = \frac{1}{\cos z} h_3(r, z) , \quad (63)$$

they are simplified

$$\begin{aligned} a h_2 &= \cos^2 z \left(+\sigma - \frac{\partial}{\partial z} \right) h_1 , \\ -b_- h_1 + a_+ h_3 &= \sigma h_2 , \\ b h_2 &= \cos^2 z \left(-\sigma - \frac{\partial}{\partial z} \right) h_3 . \end{aligned} \quad (64)$$

Let us introduce new variables

$$b_- h_1 = \bar{h}_1 , \quad a_+ h_3 = \bar{h}_3 ; \quad (65)$$

from (64) it follows

$$\begin{aligned} b_- a h_2 &= \cos^2 z \left(\sigma - \frac{\partial}{\partial z} \right) \bar{h}_1 , \\ \bar{h}_3 - \bar{h}_1 &= \sigma h_2 , \\ a_+ b h_2 &= \cos^2 z \left(-\sigma - \frac{\partial}{\partial z} \right) \bar{h}_3 . \end{aligned} \quad (66)$$

Note that first and third equations contain one the same second order operator

$$b_- a = a_+ b = \frac{1}{2} \left(-\frac{\partial^2}{\partial r^2} - \frac{\cos r}{\sin r} \frac{\partial}{\partial r} + \frac{m^2}{\sin^2 r} \right) = \Delta . \quad (67)$$

First, let us consider the case $\sigma \neq 0$. Equating the right-hand sides of the first and third equations in (66), we get

$$\sigma(\bar{h}_1 + \bar{h}_3) = -\frac{\partial}{\partial z}(\bar{h}_3 - \bar{h}_1) = -\sigma \frac{\partial}{\partial z} h_2 ; \quad (68)$$

that is

$$\bar{h}_3 + \bar{h}_1 = -\frac{\partial}{\partial z} h_2 , \quad \bar{h}_3 - \bar{h}_1 = \sigma h_2 .$$

Thus, we arrive at expression for \bar{h}_1 and \bar{h}_3 through h_2

$$\bar{h}_3 = \frac{1}{2}(+\sigma - \frac{\partial}{\partial z})h_2 , \quad \bar{h}_1 = \frac{1}{2}(-\sigma - \frac{\partial}{\partial z})h_2 . \quad (69)$$

In turn, substituting (69) into (66) we obtain one the same second order equation for h_2

$$\begin{aligned} b_- a h_2 &= \cos^2 z (\sigma - \frac{\partial}{\partial z}) \frac{1}{2} (-\sigma - \frac{\partial}{\partial z}) h_2 , \\ a_+ b h_2 &= \cos^2 z (-\sigma - \frac{\partial}{\partial z}) \frac{1}{2} (+\sigma - \frac{\partial}{\partial z}) h_2 . \end{aligned} \quad (70)$$

The variables in (70) are separated straightforwardly

$$\begin{aligned} h_2(r, z) &= h_2(r) h_2(z) , \\ \frac{1}{h_2(r)} (2b_- a) h_2(r) &= \frac{1}{h_2(z)} \cos^2 z (\frac{d^2}{dz^2} - \sigma^2) h_2(z) = \Lambda , \end{aligned}$$

from whence it follows separated differential equations

$$(2b_- a) h_2(r) = \Lambda h_2(r) , \quad (71)$$

$$(\frac{d^2}{dz^2} - \sigma^2) h_2(z) = \frac{\Lambda}{\cos^2 z} h_2(z) . \quad (72)$$

Similar results are valid for functions e_i and φ_i :

$$\begin{aligned} (2b_- a) e_2(r) &= \Lambda e_2(r) , \\ (\frac{d^2}{dz^2} - \sigma^2) e_2(z) &= \frac{\Lambda}{\cos^2 z} e_2(z) , \\ \bar{e}_1 &= \frac{1}{2}(-\sigma - \frac{\partial}{\partial z}) e_2 , \quad \bar{e}_3 = \frac{1}{2}(+\sigma - \frac{\partial}{\partial z}) e_2 ; \end{aligned} \quad (73)$$

$$\begin{aligned} (2b_- a) \varphi_2(r) &= \Lambda \varphi_2(r) , \\ (\frac{d^2}{dz^2} - \sigma^2) \varphi_2(z) &= \frac{\Lambda}{\cos^2 z} \varphi_2(z) , \\ \bar{\varphi}_1 &= \frac{1}{2}(-\sigma - \frac{\partial}{\partial z}) \varphi_2 , \quad \bar{\varphi}_3 = \frac{1}{2}(+\sigma - \frac{\partial}{\partial z}) \varphi_2 . \end{aligned} \quad (74)$$

Now, let us turn to the system (66) when $\sigma = 0$; it gives

$$\begin{aligned}\bar{h}_3 &= \bar{h}_1 = \bar{h} , \\ b_- a \, h_2 &= -\cos^2 z \frac{\partial}{\partial z} \bar{h} , \\ a_+ b \, h_2 &= -\cos^2 z \frac{\partial}{\partial z} \bar{h} .\end{aligned}\tag{75}$$

Just these relations were used above starting with (53).

Let us construct solutions of eqs. (74):

$$\begin{aligned}(2b_- a) \, \varphi_2(r) &= \Lambda \, \varphi_2(r) , \\ \left(\frac{d^2}{dz^2} - \sigma^2\right) \varphi_2(z) &= \frac{\Lambda}{\cos^2 z} \, \varphi_2(z) , \\ \bar{\varphi}_1 &= \frac{1}{2} \left(-\sigma - \frac{\partial}{\partial z}\right) \varphi_2 , \quad \bar{\varphi}_3 = \frac{1}{2} \left(+\sigma - \frac{\partial}{\partial z}\right) \varphi_2 .\end{aligned}\tag{76}$$

In the radial equation

$$(2b_- a) \, \varphi_2(r) = \Lambda \, \varphi_2(r) ,$$

or

$$\left(\frac{d^2}{dr^2} + \frac{\cos r}{\sin r} \frac{d}{dr} - \frac{m^2}{\sin^2 r} + \Lambda\right) \varphi_2(r) = 0 ;\tag{77}$$

let us introduce a new variable $1 - \cos r = 2x$, $x \in [0, 1]$:

$$x(1-x) \frac{d^2 \varphi_2}{dx^2} + (1-2x) \frac{d\varphi_2}{dx} + \left(\Lambda - \frac{1}{4} \frac{m^2}{x} - \frac{1}{4} \frac{m^2}{1-x}\right) \varphi_2 = 0\tag{78}$$

and make a substitution $\varphi_2 = x^a (1-x)^b F_2$; thus we arrive at

$$\begin{aligned}& x(1-x) \frac{d^2 F_2}{dx^2} + [2a+1 - (2a+2b+2)x] \frac{dF_2}{dx} + \\ & + \left[-(a+b)(a+b+1) + \Lambda + \frac{1}{4} \frac{4a^2 - m^2}{x} + \frac{1}{4} \frac{4b^2 - m^2}{1-x}\right] F_2 = 0 .\end{aligned}\tag{79}$$

At a, b taken according to

$$a = \pm \frac{|m|}{2}, \quad b = \pm \frac{|m|}{2},\tag{80}$$

eq. (79) becomes simpler

$$x(1-x) \frac{d^2 F_2}{dx^2} + [2a+1 - (2a+2b+2)x] \frac{dF_2}{dx} - [(a+b)(a+b+1) - \Lambda] F_2 = 0 \quad (81)$$

it represents a hypergeometric equations [7] with parameters

$$\begin{aligned} \alpha &= a+b + \frac{1}{2} - \frac{1}{2} \sqrt{1+4\Lambda}, \\ \beta &= a+b + \frac{1}{2} + \frac{1}{2} \sqrt{1+4\Lambda}, \quad \gamma = 2a+1. \end{aligned} \quad (82)$$

By physical reason for a, b we take positive values

$$a = +\frac{|m|}{2}, \quad b = +\frac{|m|}{2}; \quad (83)$$

so the radial function looks as

$$\varphi_2(r) = \left(\sin \frac{r}{2}\right)^{+|m|} \left(\cos \frac{r}{2}\right)^{+|m|} F(\alpha, \beta, \gamma; \sin^2 \frac{r}{2}); \quad (84)$$

these solutions vanish at the points $r = 0, +\pi$. To have polynomials one should impose the known condition $\alpha = -n_r$, so we get a quantization rule

$$+ \frac{\sqrt{1+4\Lambda}}{2} = n_r + |m| + \frac{1}{2}; \quad (85)$$

corresponding solutions are defined according to

$$\begin{aligned} \varphi_2 &= \left(\sin \frac{r}{2}\right)^{+|m|} \left(\cos \frac{r}{2}\right)^{+|m|} \times \\ &\times F(-n, 2|m|+1+n, |m|+1; -\sin^2 \frac{r}{2}). \end{aligned} \quad (86)$$

Now, let us solve equation (76) in variable z

$$\left(\frac{d^2}{dz^2} - \sigma^2\right) \varphi_2(z) = \frac{\Lambda}{\cos^2 z} \varphi_2(z), \quad -\sigma^2 = \epsilon^2 - M^2. \quad (87)$$

A first step is to introduce a new variable (which distinguish between conjugated point $+z$ and $-z$ of spherical space)

$$y = \frac{1+i \tan z}{2}, \quad 1-y = \frac{1-i \tan z}{2}; \quad (88)$$

if $z \in [-\pi/2, +\pi/2]$, the variable y belongs to a vertical line in the complex plane

$$y = \left(\frac{1}{2} - i\infty, \frac{1}{2} + i\infty\right). \quad (89)$$

Allowing for

$$\begin{aligned} \frac{d}{dz} &= \frac{i}{2} \frac{1}{\cos^2 z} \frac{d}{dy} = 2iy(1-y) \frac{d}{dy}, \\ \frac{\Lambda}{\cos^2 z} &= 4\Lambda y(1-y), \end{aligned} \quad (90)$$

eq. (87) reduces to

$$\left(y(1-y) \frac{d^2}{dy^2} + (1-2y) \frac{d}{dy} + \Lambda - \frac{\epsilon^2 - M^2}{4y(1-y)}\right) \varphi_2 = 0. \quad (91)$$

In the region $y \sim 0$, eq. (91) becomes simpler

$$\begin{aligned} \left(y \frac{d^2}{dy^2} + \frac{d}{dy} - \frac{\epsilon^2 - M^2}{4y}\right) \varphi_2 &= 0, \quad \varphi_2 \sim y^a, \\ a(a-1) + a - \frac{\epsilon^2 - M^2}{4} &= 0, \quad a = \pm \frac{\sqrt{\epsilon^2 - M^2}}{2}. \end{aligned} \quad (92)$$

In the region $y \sim 1$, eq. (91) becomes simpler as well

$$\begin{aligned} \left((1-y) \frac{d^2}{dy^2} - \frac{d}{dy} - \frac{\epsilon^2 - M^2}{4(1-y)}\right) \varphi_2 &= 0, \quad \varphi_2 \sim (1-y)^b, \\ b(b-1) + b - \frac{\epsilon^2 - M^2}{4} &= 0, \quad b = \pm \frac{\sqrt{\epsilon^2 - M^2}}{2}. \end{aligned} \quad (93)$$

Searching solutions in the form $\varphi_2(y) = y^a(1-y)^b F(y)$ for $F(y)$ we have

$$\begin{aligned} &y(1-y)F'' + F'[(2a+1) - y(2a+2b+2)]F' + \\ &+ \left[-(a+b)(a+b+1) + \Lambda + \frac{1}{y} \left(a^2 - \frac{\epsilon^2 - M^2}{4}\right) + \right. \\ &\quad \left. + \frac{1}{1-y} \left(a^2 - \frac{\epsilon^2 - M^2}{4}\right)\right] F = 0. \end{aligned} \quad (94)$$

Let it be

$$\begin{aligned} a &= \pm \frac{\sqrt{\epsilon^2 - M^2}}{2}, \quad b = \pm \frac{\sqrt{\epsilon^2 - M^2}}{2}, \\ \varphi_2 &= \left(\frac{1+i \tan z}{2}\right)^a \left(\frac{1-i \tan z}{2}\right)^b F; \end{aligned} \quad (95)$$

there are four possibilities depending on a, b

$$\begin{aligned}
a = b = -\frac{\sqrt{\epsilon^2 - M^2}}{2}, \quad \varphi_2 &\sim \cos^{-2a} z F(z) ; \\
a = b = +\frac{\sqrt{\epsilon^2 - M^2}}{2}, \quad \varphi_2 &\sim \cos^{-2a} z F(z) ; \\
a = -b, \quad \varphi_2 &\sim +e^{+2iaz} F(z) ; \\
a = -b, \quad \varphi_2 &\sim -e^{-2iaz} F(z) .
\end{aligned} \tag{96}$$

As relations (95) hold, eq. (94) takes the form

$$\begin{aligned}
y(1-y)F'' + F'[(2a+1) - y(2a+2b+2)]F' - \\
- [(a+b)(a+b+1) - \Lambda] F = 0 ,
\end{aligned} \tag{97}$$

which can be recognized as a hypergeometric equation [7]

$$y(1-y) F + [\gamma - (\alpha + \beta + 1)y] F' - \alpha\beta F = 0 , \tag{98}$$

$$\begin{aligned}
\gamma = (2a+1) , \quad \alpha = a+b + \frac{1}{2} + \frac{\sqrt{4\Lambda+1}}{2} , \\
\beta = a+b + \frac{1}{2} - \frac{\sqrt{4\Lambda+1}}{2} .
\end{aligned} \tag{99}$$

In this point we should notice that the spectrum for Λ has been found (see (85)) from analyzing the differential equation in the variable r , therefore now to produce a spectrum for energy one must consider the cases with $a = b$.

There arise two possibilities.

The first:

$$\begin{aligned}
2a = 2b = \sqrt{\epsilon^2 - M^2} , \quad \beta = -n_z , \\
-\sqrt{\epsilon^2 - M^2} = n_z + \frac{1}{2} - \frac{\sqrt{4\Lambda+1}}{2} < 0 , \\
\varphi_2 \sim (\cos z)^{-\sqrt{\epsilon^2 - M^2}} P_n\left(\frac{e^{iz}}{2\cos z}\right) , \quad y = \frac{1+i\tan z}{2} = \frac{e^{iz}}{2\cos z} ,
\end{aligned} \tag{100}$$

because those solutions tends to infinity at $z = \pm\pi$ they cannot describe physical bound states.

The second:

$$\begin{aligned}
-2a = -2b = +\sqrt{\epsilon^2 - M^2} , \quad \alpha = -n_z , \\
+\sqrt{\epsilon^2 - M^2} = n_z + \frac{1}{2} + \frac{\sqrt{4\Lambda+1}}{2} > 0 , \\
\varphi_2 \sim (\cos z)^{+\sqrt{\epsilon^2 - M^2}} P_n\left(\frac{e^{iz}}{2\cos z}\right) , \quad y = \frac{1+i\tan z}{2} = \frac{e^{iz}}{2\cos z} .
\end{aligned} \tag{101}$$

These solutions are finite at the points $z = \pm\pi/2$ and they describe bound states.

In the formula for $\sqrt{\epsilon^2 - M^2}$ (101) one must take into account the quantization rule Λ in (85) – thus we arrive at the formula determining values of energy by two discrete quantum numbers.

$$+ \sqrt{\epsilon^2 - M^2} = n_z + n_r + |m| + 1 ; \quad (102)$$

remember that these formulas concern the non-zero values for helicity operator $\sigma = \pm i\sqrt{\epsilon^2 - M^2}$.

It remains to specify the energy spectrum for states with $\sigma = 0$ which are determined by the equations

$$\begin{aligned} \left(-2\Delta + \frac{\partial}{\partial z} \cos^2 z \frac{\partial}{\partial z} + (\epsilon^2 - M^2) \cos^2 z \right) \bar{\varphi} &= 0 , \\ \bar{\varphi}(r, z) &= \bar{\varphi}(r) \bar{\varphi}(z) , \\ \frac{1}{\bar{\varphi}(r)} (2\Delta) \bar{\varphi}(r) &= \Lambda , \\ \frac{1}{\bar{\varphi}(z)} \left(\frac{d}{dz} \cos^2 z \frac{d}{dz} + (\epsilon^2 - M^2) \cos^2 z \right) \bar{\varphi}(z) &= \Lambda . \end{aligned} \quad (103)$$

Equation in the variable r has been solved above. The equation in z variable

$$\left(\frac{d^2}{dz^2} - 2 \frac{\sin z}{\cos z} \frac{d}{dz} + \epsilon^2 - M^2 - \frac{\Lambda}{\cos^2 z} \right) \varphi(z) = 0$$

with the use of substitution $\varphi = \frac{1}{\cos z} f(z)$ reduces to

$$\frac{d^2 f}{dz^2} + \left(\epsilon^2 - M^2 + 1 - \frac{\Lambda}{\cos^2 z} \right) f(z) = 0 . \quad (104)$$

It coincides with (87)

$$\frac{d^2 \varphi_2}{dz^2} + \left(\epsilon^2 - M^2 - \frac{\Lambda}{\cos^2 z} \right) \varphi_2(z) = 0 ,$$

with one formal change

$$\epsilon^2 - M^2 \rightarrow \epsilon^2 - M^2 + 1 .$$

Therefore, solutions of (104) are written straightforwardly

$$f = \left(\frac{1 + i \tan z}{2} \right)^a \left(\frac{1 - i \tan z}{2} \right)^b F \left(\alpha, \beta, \gamma; \frac{1 + i \tan z}{2} \right) ,$$

where F stand for a hypergeometric function [7] with parameters

$$\alpha = a + b + \frac{1}{2} + \frac{\sqrt{4\Lambda + 1}}{2},$$

$$\beta = a + b + \frac{1}{2} - \frac{\sqrt{4\Lambda + 1}}{2}, \gamma = (2a + 1).$$

To bound states correspond a, b defined as

$$a = b = -\frac{\sqrt{\epsilon^2 - M^2 + 1}}{2}.$$

The quantization rule $\alpha = -n_z$ gives

$$+\sqrt{\epsilon^2 - M^2 + 1} = n_z + \frac{1}{2} + \frac{\sqrt{4\Lambda + 1}}{2} > 0.$$

Thus, allowing for quantization for Λ (85) we get a formulas for energy levels

$$+\sqrt{\epsilon^2 - M^2 + 1} = n_z + n_r + |m| + 1; \quad (105)$$

it refers to the case of $\sigma = 0$.

Conclusion:

Let us summarize result.

Spin 1 particle is investigated in 3-dimensional curved space of constant positive curvature. An extended helicity operator is defined and the variables are separated in a tetrad-based 10-dimensional Duffin-Kemmer equation in quasi cylindrical coordinates. The problem is solved exactly in hypergeometric functions, the energy spectrum determined by three discrete quantum numbers is obtained. Transition to a massless case of electromagnetic field is performed.

The given problem can represent some interest as an exactly solvable model for describing composite systems (particles) of spin 1 or electromagnetic fields in the non-trivial space-time background, modeling the presence of a finite 3-dimensional box.

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